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Long-wavelength instability of a line plume

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The stability of a steady, laminar, immiscible, dense line plume falling through a fluid whose horizontal extent is much larger than the plume thickness is investigated. It is shown analytically that the primary flow is linearly unstable in the limit of large wavelength and, if surface tension is negligible, is also unstable in the limit of small Reynolds number. A rich variety of modes of perturbation exist, characterized by a range of balances on different lengthscales between diffusion, advection and propagation of vorticity.

1. Introduction

We consider the stability of the two-dimensional flow due to the steady fall of a narrow, laminar line plume through a fluid of large horizontal extent. The two fluids have different densities and viscosities and are considered to be incompressible and immiscible. We concentrate on long-wavelength disturbances to the primary flow noting that the short-wavelength instability (Hooper & Boyd 1983) has been shown to have a negligibly small growth rate (Hinch 1984). We also note that instability has been experimentally observed in the axisymmetric case at wavelengths much longer than the plume thickness (Huppert *et al.* 1986).

Previous authors (Yih 1963; Yih 1967; Hickox 1971; Joseph, Renardy & Renardy 1984) have considered the stability at long wavelengths of other two-fluid Couette or Poiseuille flows and have shown that interfacial instability persists at arbitrarily small Reynolds number. However, their basic flows were bounded by rigid walls close to the interface and their analyses break down in the case considered here – that of boundaries distant from the interface. The stability of plane Couette flow of two semi-infinite layers has been analysed by Hooper & Boyd (1983). They found that the flow was unstable to short-wavelength disturbances but stable to long wavelengths. Hooper (1985) has extended this work to show that Couette flow of a semi-infinite layer and a finite layer bounded by a rigid wall is unstable to long wavelengths if the finite layer has the greater viscosity.

The release of a buoyant fluid into a large reservoir of stagnant fluid occurs naturally in replenished magma chambers and has been investigated experimentally by Huppert *et al.* (1986). Our results are relevant in this context and to any other similar 'nearly unbounded' two-fluid flows. The analysis will show that both the distant nature of the boundaries and the lengthscale imposed by the plume width play an important role in the stability problem.

The basic flow is described in §2. The linear stability problem for this flow is formulated in §3. The resulting differential equation and boundary conditions for the stream function constitute an eigenvalue problem for the complex wave speed. Solutions are presented in §§4 and 5 for both varicose and meandering disturbances in each of the limits of small wavelength and of small Reynolds number showing that instability of some form will always occur. A description of the physical mechanisms of propagation and growth of disturbances at small Reynolds number is contained within the analysis of §5. The results for this problem and the analogous problem for the stability of a three-dimensional axisymmetric plume are discussed in §6.

2. The primary flow

The basic-flow configuration is sketched in figure 1. A dense fluid in $-a \leqslant r \leqslant a$ of density ρ_i and kinematic viscosity ν_i falls through a fluid of density ρ_o and kinematic viscosity ν_o, driven by the density difference. The flow is bounded by rigid walls at $r = \pm A$. We suppose that the flux of dense fluid in the plume is $2Q$ and that the modified pressure gradient, $-g\rho + \partial P/\partial x$, is $\rho_i K_i$ within the plume and $\rho_o K_o$ outside the plume. We assume that there is no variation in time or in the axial direction x, and seek the steady fluid velocity $U(r)$ and the plume width $2a$.

The release of dense fluid into $-A \leqslant r \leqslant A$ will initially produce a time-dependent and axially varying flow. This flow, however, will approach a steady, parallel, shear flow as vorticity diffuses to the walls. We concentrate on the simple case of the stability of the ultimate steady state in order to elucidate general instability mechanisms applicable to more complicated flows.

Application of the Navier–Stokes equations, the no-slip boundary condition at the rigid wall and continuity of velocity and tangential stress at the interface, together with the use of symmetry, gives

$$\nu U'' = K, \quad U(A) = 0, \quad [U(a)]^+_- = 0, \tag{2.1 a–c}$$

$$[\rho\nu U'(a)]^+_- = 0, \quad \int_0^a U \, \mathrm{d}r = Q \quad \text{and} \quad U'(0) = 0, \tag{2.1 d–f}$$

where $[\ ]^+_-$ denotes the interfacial jump from values at $r = a_-$ to $r = a_+$.

The solution to these equations is

$$U(r) = \sigma\left[(m^{-1}-k)(r-A) + \frac{k}{2a}(r^2-A^2)\right] \quad (r > a), \tag{2.2a}$$

$$U(r) = \sigma\left[(m^{-1}-k)(a-A) + \frac{k}{2a}(a^2-A^2) + \frac{\beta}{2a}(r^2-a^2)\right] \quad (r < a), \tag{2.2b}$$

where

$$\beta = \frac{\nu_o}{\nu_i}, \quad m = \frac{\rho_o}{\rho_i}, \quad k = \frac{K_o}{K_i}, \quad \sigma = \frac{K_i a}{\nu_o}. \tag{2.3}$$

We can also show that

$$K_i(1-mk) = -g', \tag{2.4}$$

where

$$g' = g\left[\frac{\rho_i - \rho_o}{\rho_i}\right]. \tag{2.5}$$

The problem is then closed by specification of the remaining unknown k. One natural condition to impose is that the pressure gradient in the large body of external fluid is hydrostatic. This requires $k = 0$. Alternatively we could desire zero net flux across a cross-section for which we need only the leading-order condition in $a/A \ll 1$. This is $k = -(3/2m)(a/A)$.

The plume thickness is determined by the requirement that the flux of buoyant fluid be $2Q$, as specified by (2.1 e). In many cases of natural interest the densities of

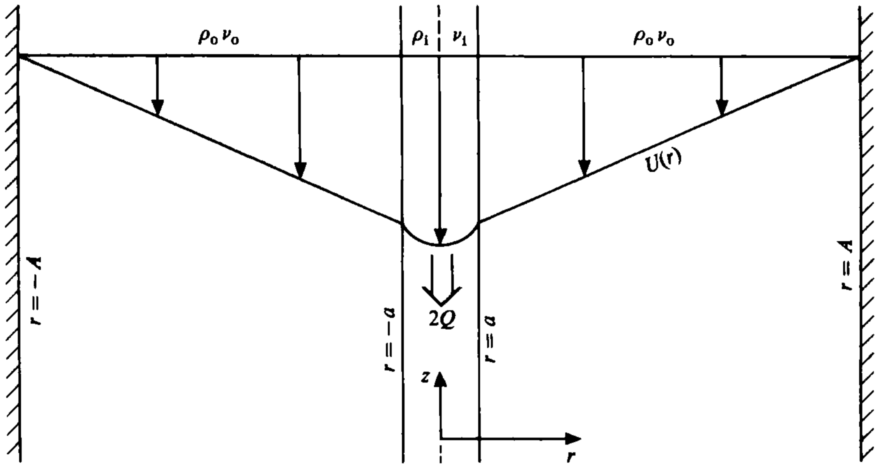

FIGURE 1. Definition sketch for the basic flow. A dense fluid falls in a line plume of thickness $2a$ through another fluid bounded by parallel rigid walls a distance $2A$ apart.

the two fluids will be nearly equal and $m \approx 1$. We put $m = 1$ for simplicity and use (2.2)–(2.5) to obtain

$$(1-k)\left[\frac{A}{a}-1\right]+\tfrac{1}{2}k\left[\frac{A^2}{a^2}-1\right]+\tfrac{1}{3}\beta = \frac{Q\nu_0}{a^3 g'}(1-k). \tag{2.6}$$

If the exterior pressure gradient is hydrostatic ($k = 0$), and if $A^3 \gg Q\nu_0/g'$ and thus $a/A \ll 1$, then (2.6) becomes

$$a \approx \left[\frac{Q\nu_0}{Ag'}\right]^{\frac{1}{2}}. \tag{2.7}$$

Alternatively, if a return flow ensures that there is no net volume flux ($k = -3a/2A$), the same approximations yield

$$a \approx 2\left[\frac{Q\nu_0}{Ag'}\right]^{\frac{1}{2}}. \tag{2.8}$$

Note the surprising feature that even a small return flow can double the plume width produced by a given buoyant flux Q. The form of (2.7) and (2.8), though not the numerical coefficients, could be deduced by observing that in a steady state the buoyancy force, $O(\rho g' a)$, acting on the plume must be balanced by the shear stress in the outer fluid, $O(\mu_0 Q/aA)$.

3. The differential system governing stability

In this section we allow a small perturbation velocity $\boldsymbol{u} = (u, v)$ to disturb the basic flow $\boldsymbol{U} = (U, 0)$ described in §2.

We first make the problem dimensionless by scaling all lengths by a, velocities by σa, densities by ρ_0 and viscosities by ν_0. We define a Reynolds number by

$$R = \frac{\sigma a^2}{\nu_0} \tag{3.1a}$$

and a capillary number by

$$\Gamma = \frac{\gamma}{\rho_0 \nu_0 \sigma a}, \tag{3.1b}$$

where γ is the coefficient of interfacial tension between the two liquids. We change to a frame of reference in which the interface is stationary so that in dimensionless form the basic flow described in §2 becomes

$$\bar{U} = (m^{-1}-k)(r-1)+\tfrac{1}{2}k(r^2-1), \quad \rho = 1, \quad \nu = 1, \quad (r > 1), \tag{3.2a}$$

$$\bar{U} = \tfrac{1}{2}\beta(r^2-1), \quad \rho = m^{-1}, \quad \nu = \beta^{-1}, \quad (r < 1). \tag{3.2b}$$

We look for normal modes of perturbation to this flow in which all quantities are products of $e^{i\alpha(x-ct)}$ and a function of r. The fluids are considered to be incompressible and so $\boldsymbol{u}$ can be represented by

$$\boldsymbol{u} = (\mathrm{D}\psi, \ -\mathrm{i}\alpha\psi)\, e^{i\alpha(x-ct)}, \tag{3.3}$$

where $\mathrm{D} \equiv \mathrm{d}/\mathrm{d}r$. Substitution into the curl of the Navier–Stokes equation, linearized about the basic state, leads to the well-known Orr–Somerfeld equation

$$\nu(\mathrm{D}^2-\alpha^2)^2\psi = \mathrm{i}\alpha R[(\bar{U}-c)(\mathrm{D}^2-\alpha^2)\psi-\bar{U}''\psi]. \tag{3.4a}$$

The symmetry of the basic-flow geometry leads to the invariance of (3.4a) under $r \rightarrow -r$ and allows us to seek even or odd solutions for ψ, corresponding to meandering or varicose perturbations of the interface respectively. Thus either

$$\mathrm{D}\psi(0) = \mathrm{D}^3\psi(0) = 0 \quad \text{(meanders only)} \tag{3.4b}$$

or

$$\psi(0) = \mathrm{D}^2\psi(0) = 0 \quad \text{(varicose only)}. \tag{3.4c}$$

The boundary conditions at the rigid wall require

$$\psi(A) = \mathrm{D}\psi(A) = 0. \tag{3.4d}$$

The remaining boundary conditions arise from linearized equations for the continuity of the two components of velocity and stress at the interface. The continuity of radial velocity, axial velocity and tangential stress respectively imply that

$$[\psi]^+_- = 0. \tag{3.4e}$$

$$\left[\mathrm{D}\psi+\frac{\psi}{c}\bar{U}'\right]^+_- = 0, \tag{3.4f}$$

$$\left[\rho\nu\left((\mathrm{D}^2+\alpha^2)\psi+\frac{\psi}{c}\bar{U}''\right)\right]^+_- = 0. \tag{3.4g}$$

The jump in the normal component of hydrodynamic stress must be balanced by surface tension. After some manipulation we obtain

$$[\rho\nu(\mathrm{D}^2-3\alpha^2)\mathrm{D}\psi]^+_- = -\frac{\mathrm{i}\Gamma\alpha^3\psi}{c}-\mathrm{i}\alpha R(c\mathrm{D}\psi+\bar{U}'\psi)(1-m^{-1}), \tag{3.4h}$$

where $[\ ]^+_-$ denotes the jump from values at $r = 1_-$ to values at $r = 1_+$. The derivation is given in Yih (1967), though (3.4g) differs from the analogous equations in previous work owing to the driving pressure gradient in the descending fluid.

Equations (3.4a–h) form an eigenvalue problem for c as a function of the dimensionless parameters R, Γ, m, β, A and α and there is instability if c has a positive imaginary part for some α.

With six independent parameters to consider, it is necessary to make approximations in order to make further progress. Previous authors (Hickox 1971; Joseph *et al.* 1984) have obtained results for axisymmetric plumes which are valid as $\alpha \rightarrow 0$ with

A fixed and comparable with the plume width. Their results break down in the case we are interested in here: that in which A is very large and the variations on lengthscales α^{-1}, implicit in the operator $\mathrm{D}^2-\alpha^2$, are important. For the sake of definiteness in the joint limit $A\rightarrow\infty$, $\alpha\rightarrow 0$, we formally put $A=\infty$ in (3.6*d*) and hence, from §2, $k=0$ in (3.2*a*).

The removal of the boundary conditions (3.6*d*) to infinity is equivalent to the selection of the exponentially decaying solutions of (3.4*a*); the growing solutions will have an exponentially small coefficient and will make a negligible contribution to the jump conditions (3.4*e–h*) and the growth rate c. The approximation of the basic-flow profile is valid on the lengthscale α^{-1} of the disturbance flow field when $\alpha A \gg 1$ as considered here. We note, also, that the time taken to set up the steady state whose stability is under consideration will increase with A. However, if our analysis predicts instability then we know that a line plume will ultimately become unstable. At sufficiently large times the basic flow may be treated as quasi-steady and our analysis applied. Moreover, even if the time taken to attain a steady state is large, we expect that the instability mechanisms for the steady state will also apply to the transient time-dependent flow.

As commented earlier, the densities of the two fluids will often be nearly equal and thus $m\approx 1$. It is possible to retain a general value of m throughout the following analysis but it is found that this merely complicates the algebra without changing the number or stability of any of the modes of perturbation. The approximation $m=1$, corresponding to the Boussinesq approximation, will thus be made for simplicity, the errors involved being $O(m-1)$. The intention of this approximation is to ignore the unnecessary complications of the inertial effects of the density difference between the fluids; the buoyancy effects of the density difference drive the basic flow and cannot be ignored. The imposition of $m=1$ should, therefore, only be made in (3.2) and (3.4) and should not be taken to imply anything about the strength of the basic flow or the values of σ, R and Γ.

In §4 we consider the limit $\alpha\rightarrow 0$ and in §5 we consider the limit $R\rightarrow 0$.

4. Solutions valid as $\alpha\rightarrow 0$

In this section we present solutions to (3.2) and (3.4) that are valid as $\alpha\rightarrow 0$ under the approximations $A=0$, $k=0$, $m=1$. Specifically, we solve

$$\nu(\mathrm{D}^2-\alpha^2)^2\psi = \mathrm{i}\alpha R[(\tfrac{1}{2}\beta(r^2-1)-c)(\mathrm{D}^2-\alpha^2)\psi-\beta\psi] \quad (r<1), \tag{4.1a}$$

$$(\mathrm{D}^2-\alpha^2)^2\psi = \mathrm{i}\alpha R(r-1-c)(\mathrm{D}^2-\alpha^2)\psi \quad (r>1), \tag{4.1b}$$

$$\psi\rightarrow 0 \quad \text{as} \quad r\rightarrow\infty, \tag{4.2}$$

$$\psi(0)=\mathrm{D}^2\psi(0)=0 \quad \text{(varicose only)}, \tag{4.3a}$$

$$\mathrm{D}\psi(0)=\mathrm{D}^3\psi(0)=0 \quad \text{(meanders only)}, \tag{4.3b}$$

$$[\psi]^{+}_{-}=0, \tag{4.4a}$$

$$[c\mathrm{D}\psi]^{+}_{-}+(1-\beta)\psi=0, \tag{4.4b}$$

$$[c\nu(\mathrm{D}^2+\alpha^2)\psi]^{+}_{-}-\psi=0, \tag{4.4c}$$

$$[\nu(\mathrm{D}^2-3\alpha^2)\mathrm{D}\psi]^{+}_{-}=-\frac{\mathrm{i}\Gamma\alpha^3}{c}\psi, \tag{4.4d}$$

with $\nu=1$ in $r>1$ and $\nu=\beta^{-1}$ in $r<1$.

We expand ψ and c as asymptotic series in powers of α and solve (4.1) at successive orders. It is found that (4.1) has different asymptotic forms when $r = O(1)$, when $\alpha r = O(1)$ and possibly when $\alpha^{\frac{1}{3}} r = O(1)$. The eigenvalue c is determined by matching the different solutions for ψ and applying (4.2)–(4.4). Two possible expansions are given for meandering instability and one for varicose instability.

Surface tension will only be important if it is very large ($\Gamma = O(\alpha^3)$) and it is more realistic to assume that $\Gamma = O(1)$. As we would expect, surface tension then plays no role in these long-wavelength instabilities.

4.1. *A varicose perturbation with* $c = O(\alpha^{-\frac{1}{3}})$

Pose

$$c \sim c_{-1}\alpha^{-\frac{1}{3}} + c_0 + c_1\alpha^{\frac{1}{3}}, \tag{4.5}$$

$$\psi \sim \psi_0 + \alpha^{\frac{1}{3}}\psi_1 + \alpha^{\frac{2}{3}}\psi_2, \tag{4.6}$$

and let $q = r-1$, $s = \alpha^{\frac{1}{3}}(r-1)$, $t = \alpha(r-1)$.

Substituting into (4.1*b*), we obtain

$$\frac{\mathrm{d}^4\psi}{\mathrm{d}q^4} = -\alpha^{\frac{2}{3}}\mathrm{i}Rc_{-1}\frac{\mathrm{d}^2\psi_0}{\mathrm{d}q^2} + O(\alpha), \tag{4.7a}$$

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}s^2} - \mathrm{i}R(s-\alpha^{\frac{1}{3}}c)\right]\frac{\mathrm{d}^2\psi}{\mathrm{d}s^2} = O(\alpha^{\frac{4}{3}}), \tag{4.7b}$$

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}t^2} - 1\right]\psi = O(\alpha^2). \tag{4.7c}$$

Thus, under the ansatz (4.5) and (4.6), we see from equations (4.7) that the flow divides into three regions: on lengthscales $O(1)$ only diffusion of disturbance vorticity is important; on lengthscales $O(\alpha^{-1})$ only advection by the mean flow is important; and in a region where $r = O(\alpha^{-\frac{1}{3}})$ there is a balance between diffusion and advection.

In the region $r = O(\alpha^{-1})$, (4.7*c*) is easily solved subject to the boundary condition (4.2). Since the whole problem is homogeneous in ψ we can write, without loss of generality,

$$\psi(t) = \mathrm{e}^{-t} + O(\alpha^2), \tag{4.8}$$

In the region $r = O(\alpha^{-\frac{1}{3}})$, (4.7*b*) can be solved at leading orders to give

$$\frac{\mathrm{d}^2\psi_i}{\mathrm{d}s^2} = B_i \operatorname{Ai}[\mathrm{e}^{\pi\mathrm{i}/6}R^{\frac{1}{3}}(s-\alpha^{\frac{1}{3}}c)] + C_i \operatorname{Ai}[\mathrm{e}^{5\pi\mathrm{i}/6}R^{\frac{1}{3}}(s-\alpha^{\frac{1}{3}}c)] \quad (i = 0, 1, 2, 3). \tag{4.9}$$

Since

$$\operatorname{Ai}(\mathrm{e}^{5\pi\mathrm{i}/6}x) \sim \frac{1}{2\surd\pi}(\mathrm{e}^{5\pi\mathrm{i}/6}x)^{-\frac{1}{4}}\,\mathrm{e}^{(\surd 2/3)(1+\mathrm{i})x^{\frac{3}{2}}} \tag{4.10}$$

(Abramowitz & Stegun 1965), we must have $C_i = 0$ in order to match to the decaying solution (4.8). Then

$$\psi_i(s) = B_i I + D_i s + E_i \quad (i = 0, 1, 2, 3), \tag{4.11}$$

where

$$I(s, \alpha^{\frac{1}{3}}) = \int_s^\infty \int_v^\infty \operatorname{Ai}[R^{\frac{1}{3}}\,\mathrm{e}^{\pi\mathrm{i}/6}(u-\alpha^{\frac{1}{3}}c)]\,\mathrm{d}u\,\mathrm{d}v \tag{4.12}$$

and by matching to (4.8) $E_0 = 1$, $D_0 = 0$, $E_1 = 0$, $D_1 = 0$, $E_2 = 0$ and $D_2 = -1$. In preparation for matching inwards to $r = O(1)$ let

$$I(s, \alpha^{\frac{1}{3}}) = I_0(s) + \alpha^{\frac{1}{3}}I_1(s) + \alpha^{\frac{2}{3}}I_2(s) + O(\alpha) \tag{4.13a}$$

and

$$I_j(s) = I_j + sI_j' + \tfrac{1}{2}s^2 I_j'' + O(s^3). \tag{4.13b}$$

To complete the solution we must solve (4.1) in the region $r = O(1)$, match to (4.11) and impose the boundary conditions (4.3*a*) and (4.4). We describe in detail the leading-order derivation and sketch the result at higher orders. At leading order (4.1) becomes

$$\frac{d^4\psi_0}{dr^4} = 0. \tag{4.14}$$

We solve this subject to (4.3*a*) and match to (4.11):

$$\psi_0(r) = 1 + B_0 I_0 \quad (r > 1), \tag{4.15a}$$

$$\psi_0(r) = P_0 r + Q_0 r^3 \quad (r < 1). \tag{4.15b}$$

Use of (4.4*d*) gives $Q_0 = 0$. Equation (4.4*b*) or (4.4*c*) then implies $P_0 = 0$ and finally (4.4*a*) requires $B_0 = -I_0^{-1}$. Note that $d^2\psi_0/dr^2 = 0$ so ψ_1 and ψ_2 are also cubic in r. After matching and imposition of the boundary conditions, we find that

$$\psi_1(r) = -I_0^{-1} I_0' r \quad (r > 1 \quad \text{and} \quad r < 1), \tag{4.16a}$$

$$\psi_2(r) = -\tfrac{1}{2} I_0^{-1} I_0'' q^2 + (B_1 I_0' - I_0^{-1} I_1') q + B_2 I_0 + B_1 I_1 - I_0^{-1} I_2 \quad (r > 1), \tag{4.16b}$$

$$\psi_2(r) = P_2 r \quad (r < 1), \tag{4.16c}$$

where $B_1 = I_0^{-2}(I_1 - I_0')$. Finally c_{-1} is determined by substitution of (4.16) into (4.4*c*) and is given by

$$c_{-1} = \frac{I_0'}{I_0''}. \tag{4.17}$$

The constants B_2 and P_2 could if necessary be found from (4.4*a*, *b*).

Routine series expansion of (4.12) using (4.13) shows that $c_{-1} = R^{-\frac{1}{3}} e^{5\pi i/6} z$, where z is a root of

$$f(z) \equiv z \operatorname{Ai}(z) + \int_0^z \operatorname{Ai}(\xi)\, d\xi - \tfrac{1}{3} = 0. \tag{4.18}$$

It can be shown numerically that $f(z)$ has one positive real root in $0.7 < z < 0.8$ (Abramowitz & Stegun (1965), tables 10.11 & 10.12) and an infinite number of negative roots. Recalling that instability occurs if c has a positive imaginary part, we see that we have one unstable mode and many stable modes of perturbation.

4.2. *A meandering perturbation with* $c = O(\alpha^{-\frac{1}{3}})$

The only difference between the formulation of the stability problem for meandering perturbations and that for varicose perturbations is that the boundary conditions of (4.3*b*) are applied at the origin rather than those of (4.3*a*). We can follow the analysis of §4.1 from (4.5) to (4.13) to determine the outer flows in $r = O(\alpha^{-1})$ and in $r = O(\alpha^{-\frac{1}{3}})$. However, the analysis takes a different turn in the region $r = O(1)$.

The leading-order equations for ψ when $r = O(1)$ are still

$$\frac{d^4\psi_i}{dr^4} = 0 \quad (i = 0, 1). \tag{4.19}$$

After imposition of the boundary conditions (4.3*b*) at the origin, we have

$$\psi_i = P_i + Q_i r^2 \quad (r < 1), \tag{4.20a}$$

$$\psi_i = R_i + S_i q + T_i q^2 + U_i q^3 \quad (r > 1). \tag{4.20b}$$

The constants R_i, S_i, T_i and U_i are found by matching to (4.11), and P_i, Q_i, c_{-1} and B_i should then be obtainable from (4.4). However, we find that $U_i = 0$ $(i = 0, 1)$ and so (4.4*d*) is satisfied automatically. The system of equations is thus underspecified and after completion of any given order there will be undetermined constants. A solution can be found by looking ahead and using equations from higher orders.

At leading order we readily obtain

$$\psi_0 = 1 + B_0 I_0 \quad (r > 1 \quad \text{and} \quad r < 1), \tag{4.21}$$

where B_0 cannot be determined at this order. At $O(\alpha^{\frac{1}{3}})$, matching to (4.11) gives

$$\psi_1 = B_0 I_0' q + B_1 I_0 + B_0 I_1 \quad (r > 1). \tag{4.22}$$

At this order we have only (4.4*a–c*) to find B_0, B_1, c_{-1}, P_1 and Q_1, and hence must leave two unknown constants. We note that (4.4*c*) reads

$$(1 + B_0 I_0)\beta + 2c_{-1} Q_1 = 0 \tag{4.23}$$

and jump to $O(\alpha)$. From (4.1),

$$\frac{d^4\psi_3}{dr^4} = \frac{iR}{\nu}\left[-\bar{U}''\psi_0 - c_{-1}\frac{d^2\psi_1}{dr^2} - c_0\frac{d^2\psi_0}{dr^2}\right]. \tag{4.24}$$

Substitution from (4.21)–(4.23) shows that ψ_3 is in fact also unforced and will take the form (4.20). Then (4.4*d*) gives $U_3 = 0$, but, by matching outwards, $U_3 = \frac{1}{6}B_0 I_0'''$. If $B_0 = 0$ then we cannot simultaneously satisfy (4.4*b*) and (4.4*c*) at $O(\alpha^{\frac{1}{3}})$. We conclude, therefore, that $I_0''' = 0$, determining c_{-1}. Returning to $O(\alpha^{\frac{1}{3}})$, we find that

$$B_0 = -(I_0' c_{-1} + I_0)^{-1}, \tag{4.25a}$$

$$Q_1 = \tfrac{1}{2}\beta B_0 I_0', \tag{4.25b}$$

$$P_1 = B_1 I_0 + B_0(I_1 - \tfrac{1}{2}\beta I_0'), \tag{4.25c}$$

with B_1 still unknown. Careful consideration indicates that this method will generate higher-order solutions: (4.4*b*) and (4.4*c*) at $O(\alpha^{i/3})$ together with (4.4*d*) at $O(\alpha^{(i+2)/3})$ provide three equations for B_{i-1}, c_{i-1} and Q_i. Then (4.4*a*) at $O(\alpha^{i/3})$ gives P_i in terms of the only remaining unknown, B_i.

For our purposes, it is sufficient to know that $I_0''' = 0$. Substituting from (4.12), we find that $c_{-1} = R^{-\frac{1}{3}} e^{5\pi i/6} z$, where z is a root of $\mathrm{Ai}'(z) = 0$. All the roots of $\mathrm{Ai}'(z)$ are negative and thus all these modes of perturbation are stable.

4.3. *Perturbations with $c = O(\alpha^{-1})$*

Pose,

$$c \sim \alpha^{-1}c_{-1} + c_0 + \alpha c_1 \equiv \alpha^{-1}c_{-1} + c', \tag{4.26a}$$

$$\psi \sim \psi_0 + \alpha\psi_1 + \alpha^2\psi_2, \tag{4.26b}$$

and let $q = r-1$, $t = \alpha(r-1)$ as before. Substituting into (4.1), we obtain

$$\left[\frac{d^2}{dr^2} + \frac{iRc_{-1}}{\nu}\right]\frac{d^2\psi}{dr^2} = \frac{i\alpha R}{\nu}\left[(\bar{U} - c')\frac{d^2\psi}{dr^2} + c_{-1}\alpha\psi\right] + 2\alpha^2\frac{d^2\psi}{dr^2} + O(\alpha^3), \tag{4.27a}$$

$$\left[\frac{d^2}{dt^2} - 1\right]\psi = \frac{\alpha^2}{iR(t - \alpha c)}\left[\frac{d^2}{dt^2} - 1\right]^2\psi. \tag{4.27b}$$

Thus, under the assumptions (4.26), we have a region where $r = O(\alpha^{-1})$ in which only advection of vorticity by the mean flow is important, whereas if $r = O(1)$ there is a balance between diffusion and propagation of waves of vorticity.

In the region $r = O(\alpha^{-1})$ we can solve (4.27*b*) and use homogeneity to get

$$\psi(t) = e^{-t}. \tag{4.28}$$

When solving (4.27*a*) in $r = O(1)$ it is convenient to introduce constants $\lambda_+ = (-\mathrm{i}Rc_{-1})^{\frac{1}{2}}$ and $\lambda_- = \beta^{\frac{1}{2}}\lambda_+$, defined so $\mathrm{Re}(\lambda_\pm) > 0$. In order to be able to match to (4.28), the solution to (4.27*a*) in $r > 1$ must be

$$\psi_0(r) = B_0 + C_0 q + D_0\,\mathrm{e}^{-\lambda_+ q}. \tag{4.29}$$

where $B_0 = 1$, $C_0 = 0$. After using (4.3*a*) a varicose solution will satisfy

$$\psi_0(r) = E_0 r + F_0 \sinh(\lambda_- r) \quad (r < 1), \tag{4.30}$$

while a meandering solution satisfying (4.3*b*) will have

$$\psi_0(r) = G_0 + H_0 \cosh(\lambda_- r) \quad (r < 1). \tag{4.31}$$

We substitute (4.29) and (4.30) or (4.31) into (4.4) and attempt to solve for the unknown constants.

In the varicose case (4.30), (4.29) and (4.4) produce

$$1 + D_0 = E_0 + F_0 \sinh\lambda_-, \tag{4.32a}$$

$$\lambda_+ D_0 = E_0 + \lambda_- F_0 \cosh\lambda_-, \tag{4.32b}$$

$$\beta\lambda_+^2 D_0 = \lambda_-^2 F_0 \sinh\lambda_-, \tag{4.32c}$$

$$-\beta\lambda_+^3 D_0 = \lambda_-^3 F_0 \cosh\lambda. \tag{4.32d}$$

A non-trivial solution for D_0 and F_0 in (4.42*c, d*) would require $\coth\lambda_- = -\beta^{-\frac{1}{2}}$. This contradicts $\mathrm{Re}(\lambda_-) > 0$ and hence we must have $D_0 = F_0 = 0$. Unfortunately, substitution of these values into (4.32*a, b*) leads to $E_0 = 0$ and $E_0 = 1$. The only conclusion possible from this contradiction is that no varicose solution satisfying the asymptotic expansions of (4.26) can exist.

On the other hand the jump conditions at leading order for a meandering perturbation possess the solution $D_0 = H_0 = 0$, $G_0 = 1$, and we may proceed to the next order. Note that all the derivatives of ψ_0 are zero.

The solutions to (4.27*a*) at $O(\alpha)$ which satisfy the boundary conditions at the origin and match to (4.28) are

$$\psi_1(r) = -q + D_1\,\mathrm{e}^{-\lambda_+ q} \quad (r > 1), \tag{4.33a}$$

$$\psi_1(r) = G_1 + H_1 \cosh(\lambda_- r) \quad (r < 1). \tag{4.33b}$$

It is straightforward to deduce from the interfacial jump conditions that

$$c_{-1} = 1 - \beta, \tag{4.34}$$

$$D_1 = [(1-\beta)\lambda_+^2(1 + \beta^{-\frac{1}{2}}\coth\lambda_-)]^{-1}, \tag{4.35a}$$

$$H_1 \sinh\lambda_- = -D_1\beta^{-\frac{1}{2}}, \tag{4.35b}$$

$$G_1 = [(1-\beta)\lambda_+^2]^{-1}. \tag{4.35c}$$

Hence c_{-1} is real and there is neutral stability at this order. We may understand the propagation of the disturbance by the following argument (see figure 2). At leading order, continuity of stress can only be satisfied if there is no axial perturbation flow inside the plume. Hence, there must be an axial flow outside the plume in such a direction as to cancel the discontinuity in the basic flow at the disturbed interfacial position. The vorticity associated with this axial perturbation flow has the appropriate sign to induce propagation in the direction predicted by (4.34). Careful consideration shows that even the inclusion of the density contrast between the fluids

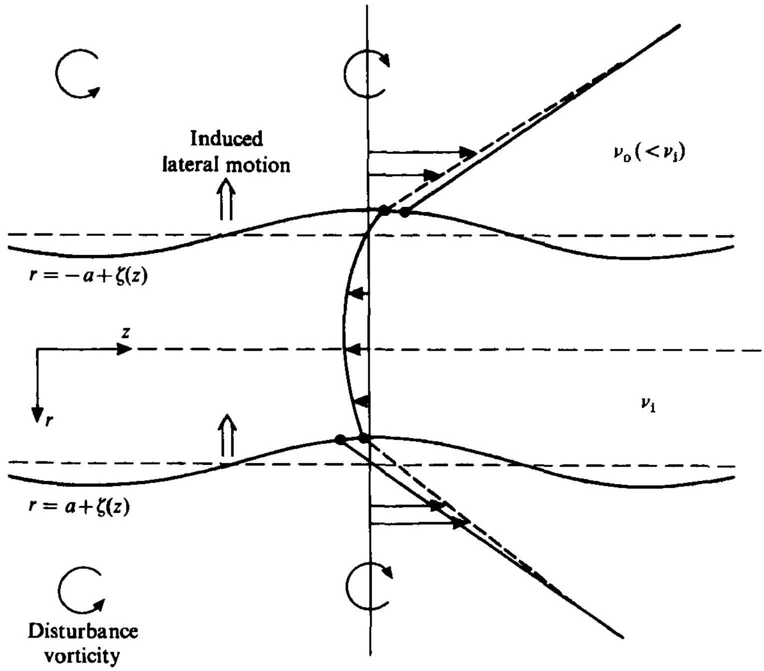

FIGURE 2. A mechanism of propagation in the limit $\alpha \to 0$. The difference between the disturbed velocity profile (shown dashed) and the undisturbed profile (shown solid) produces vorticity which causes lateral motion and propagation of the disturbance.

does not perturb c_{-1} away from a real value. We must therefore proceed to the next order.

The solution to (4.27*a*) at $O(\alpha^2)$ that satisfies the boundary conditions at the origin and matches to the flow at infinity (4.28) is given by $\psi_2 = \psi_{2C} + \psi_{2P}$ where

$$\psi_{2C} = D_2\, e^{-\lambda_+ q} \qquad (r > 1), \quad (4.36a)$$

$$\psi_{2C} = G_2 + H_2 \cosh(\lambda_- r) \qquad (r < 1), \quad (4.36b)$$

$$\psi_{2P} = (P_2 \lambda_+ q^2 + Q_2 q)\, e^{-\lambda_+ q} + R_2 q^2 \qquad (r > 1), \quad (4.36c)$$

$$\psi_{2P} = S_2\left[\frac{\lambda_- r^3}{6}\sinh\lambda_- r - \frac{5r^2}{4}\cosh\lambda_- r\right] + T_2 \frac{r}{\lambda_-}\sinh\lambda_- r + U_2 r^2 \qquad (r < 1), \quad (4.36d)$$

and

$$\left.\begin{aligned} P_2 &= \frac{D_1}{4c_{-1}}, \quad Q_2 = \frac{D_1}{4c_{-1}}(5 - 2\lambda_+ c_0), \\ R_2 &= \frac{1}{2}, \quad S_2 = -\frac{H_1 \beta}{2c_{-1}}, \\ T_2 &= -\frac{H_1\beta}{4c_{-1}}\left[\frac{17}{2} - \lambda_-^2 - \frac{2c_0 \lambda_-^2}{\beta}\right], \quad U_2 = \frac{1}{2}. \end{aligned}\right\} \quad (4.37)$$

The four unknown constants D_2, G_2, H_2 and c_0 are determined by the four interfacial jump conditions (4.4). In particular, a fair amount of laborious but routine manipulation yields the following expression for c_0:

$$c_0 = \frac{\frac{3}{4} + 1/\lambda_+ - \beta[1 + \beta^{-\frac{1}{2}}\coth(\beta^{\frac{1}{2}}\lambda_+)]}{\lambda_+[\frac{3}{2} + (1-\beta)\lambda_+[1 + \beta^{-\frac{1}{2}}\coth(\beta^{\frac{1}{2}}\lambda_+)]]}, \quad (4.38)$$

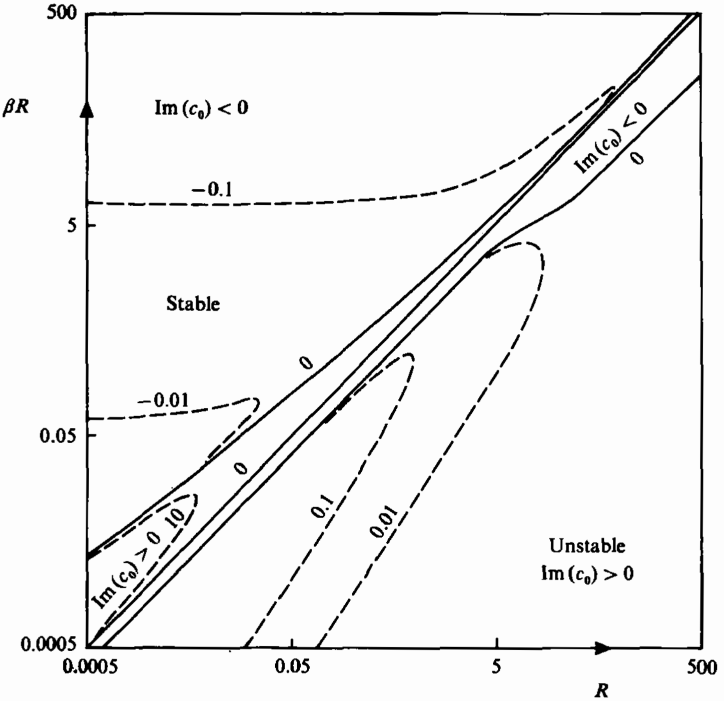

FIGURE 3. Contours of the leading-order term in the imaginary part of the growth rate in the limit $\alpha \to 0$. The flow is unstable if $\nu_0 < \beta_{crit}(R)\,\nu_i$.

where

$$\lambda_+ = \left[\frac{(1-\beta)R}{2}\right]^{\frac{1}{2}}(1-\mathrm{i}) \quad (\beta < 1) \tag{4.39a}$$

$$\lambda_+ = \left[\frac{(\beta-1)R}{2}\right]^{\frac{1}{2}}(1+\mathrm{i}) \quad (\beta > 1). \tag{4.39b}$$

A contour map of $\mathrm{Im}\,(c_0)$ plotted against R and βR is given in figure 3. The singularity at $\beta = 1$ is a consequence of the non-commutation of the limits $\beta \to 1$ and $\alpha \to 0$, and would not appear in an asymptotic expansion of c in powers of $\beta - 1$. Discounting the singularity, therefore, we see that the flow is unstable if $\nu_0 < \beta_{crit}\,\nu_i$, where $\beta_{crit}(R)$ decreases from about 6 at $R = 0.0005$ to about 0.2 at $R = 500$.

5. Long-wavelength solutions at low Reynolds number

The solutions presented in §4 all contain a region $r = O(\alpha^{-1})$ in which advective effects are important. In the limit of low Reynolds number diffusion of vorticity must be the dominant effect throughout the flow. Consideration of (4.1a) shows that advection will not be important anywhere if $R \ll \alpha^3 \ll 1$ and we would therefore expect the expansions of §4 to break down in this limit. In this section we derive an expression for c that is valid in the joint limits of $R \to 0$ and $\alpha \to 0$ subject to $R \ll \alpha^3$. We continue to use the simplifying approximations $A = \infty$, $k = 0$ and $m = 1$. We pose

$$\psi \sim \psi_0 + R\psi_1 + \dots, \tag{5.1}$$

$$c \sim c_0 + Rc_1 + \dots. \tag{5.2}$$

The leading-order problem obtained from (3.2) and (3.4) is then

$$(\mathrm{D}^2-\alpha^2)^2\psi_0 = 0, \tag{5.3a}$$

$$\psi_0 \sim 0 \quad \text{as} \quad r\to\infty, \tag{5.3b}$$

$$\psi_0(0) = \mathrm{D}^2\psi_0(0) = 0 \quad \text{(varicose only)}, \tag{5.3c}$$

$$\mathrm{D}\psi_0(0) = \mathrm{D}^3\psi_0(0) = 0 \quad \text{(meanders only)}, \tag{5.3d}$$

$$[\psi_0]_-^+ = 0, \tag{5.3e}$$

$$[c_0\,\mathrm{D}\psi_0+\tilde{U}'\psi_0]_-^+ = 0, \tag{5.3f}$$

$$[\nu(c_0(\mathrm{D}^2+\alpha^2)\,\psi_0+\tilde{U}''\psi_0)]_-^+ = 0, \tag{5.3g}$$

$$[\nu(\mathrm{D}^2-3\alpha^2)\,\mathrm{D}\psi_0]_-^+ = -\frac{\mathrm{i}\Gamma\alpha^3}{c_0}\psi_0, \tag{5.3h}$$

where in $r>1$

$$\tilde{U} = r-1, \qquad \nu = 1, \tag{5.4a}$$

and in $r<1$

$$\tilde{U} = \tfrac{1}{2}\beta(r^2-1), \quad \nu = \beta^{-1}. \tag{5.4b}$$

This perturbation scheme shows some similarity to the short-wavelength analysis of two-fluid unbounded Couette flow given by Hooper & Boyd (1983). However, the natural lengthscale imposed by the plume width means that the low-Reynolds-number and the short-wavelength limits are in fact distinct.

5.1. *Varicose perturbations*

Let $q=r-1$. The solution to (5.3*a*) subject to the boundary conditions (5.3*b*, *c*) is

$$\psi_0 = (A_0\,\alpha q+B_0)\,\mathrm{e}^{-\alpha q} \qquad (r>1), \tag{5.5a}$$

$$\psi_0 = C_0\,\alpha r\cosh(\alpha r)+D_0\sinh(\alpha r) \qquad (r<1). \tag{5.5b}$$

For brevity of notation, we write C for $\cosh\alpha$ and S for $\sinh\alpha$. When (5.5) is substituted into the interfacial-jump conditions (5.3*e–h*), we obtain a homogeneous system of equations for A_0, B_0, C_0 and D_0. This has a non-trivial solution only when the matrix of coefficients has zero determinant. This condition reduces to the equation

$$c_0 = \frac{-\left(\dfrac{1}{2\alpha}+\beta-1\right)-\tfrac{1}{2}\mathrm{i}\Gamma[S^2+(CS-\alpha)\,\beta]}{(C^2+S^2)+(CS-\alpha)\,\beta+(CS+\alpha)\,\beta^{-1}}. \tag{5.6}$$

It is routine to prove that $\operatorname{Im} c_0 < 0$ and that $\operatorname{Im} c_0$ is a monotone decreasing function of α. Thus surface tension stabilizes all wavelengths at leading order but, as we would expect, long wavelengths are only weakly stabilized. We thus concern ourselves with the long-wavelength limit and, for simplicity, approximate S by α, C by 1 and $CS-\alpha$ by $\tfrac{2}{3}\alpha^3$, and neglect higher-order terms in α. We deduce that

$$c_0 \approx -\frac{1}{2\alpha}-\tfrac{1}{2}\mathrm{i}\Gamma\alpha^2, \tag{5.7}$$

corresponding to a wave propagating in the direction of plume flow and weakly stabilized by surface tension.

At long wavelengths and small capillary numbers the stabilizing effects of surface tension may be overcome by the destabilizing influence of inertia. The first effects of a non-zero Reynolds number may be calculated by solving (3.2) at $O(R)$. ψ_1 is the

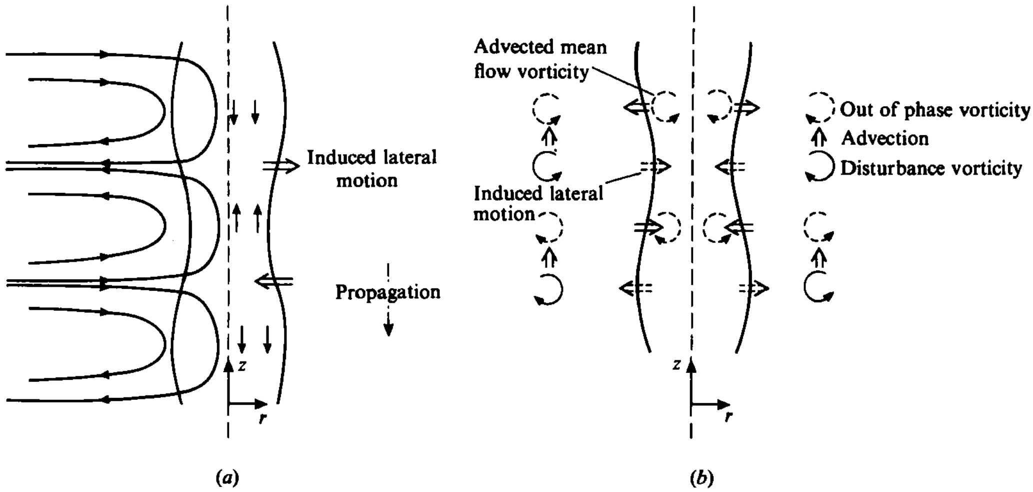

FIGURE 4. (*a*) In the limit $R \to 0$, the variation in the plume thickness causes a vertical perturbation velocity. This induces lateral interfacial motion and propagation of the disturbance downwards. (*b*) The advection of vorticity associated with the perturbation velocities in (*a*) provides a component of vorticity out of phase with the disturbance. This leads to growth of the perturbation.

sum of a particular integral forced by the advective effects of ψ_0, and a complementary solution of the same form as ψ_0 in (5.5). The four unknown constants in this complementary solution are determined from four inhomogeneous equations derived from the jump conditions at $r = 1$. The coefficients of the homogeneous part of these equations are the elements of the singular matrix of coefficients appearing at leading order. Thus, c_1 is determined by the condition that the inhomogeneous part of these equations lies in the image space of this singular matrix. A large amount of algebra yields the result that, when $\alpha \ll 1$,

$$c_1 = \frac{3\mathrm{i}}{16\alpha^3}. \tag{5.8}$$

We can thus obtain instability if $R > \frac{8}{3}\Gamma\alpha^5$ (provided still that $R \ll \alpha^3$).

We can understand the physical mechanisms of propagation and instability by the following arguments. The leading-order stream function is given by

$$\psi_0 \approx A(r\,\mathrm{e}^{-\alpha(r-1)}) \quad (r > 1), \tag{5.9a}$$

$$\psi_0 \approx A(r+\tfrac{1}{2}\alpha^2 r^3) \quad (r < 1). \tag{5.9b}$$

Where the plume is thicker than the mean thickness it falls more quickly and the perturbation velocity is downwards. Where it is thinner it falls more slowly and the perturbation velocity is upwards (see figure 4*a*). This is a simple consequence of the balance between buoyancy and surface stress. The variation in the vertical flux causes a horizontal motion that is out of phase with the interfacial displacement and in such a direction that the disturbance propagates downwards in agreement with (5.7). At $O(R)$ we must take into account the advective term in the vorticity equation. The disturbance vorticity is concentrated outside the plume and is advected upwards by the mean flow. The vorticity associated with the mean flow is advected by the flow associated with the leading-order perturbation. Both these advective effects lead to a component of disturbance vorticity at $O(R)$ that is out of phase with the interfacial perturbation and that has the appropriate sign to induce a growth of the perturbation (see figure 4*b*). This agrees with (5.8).

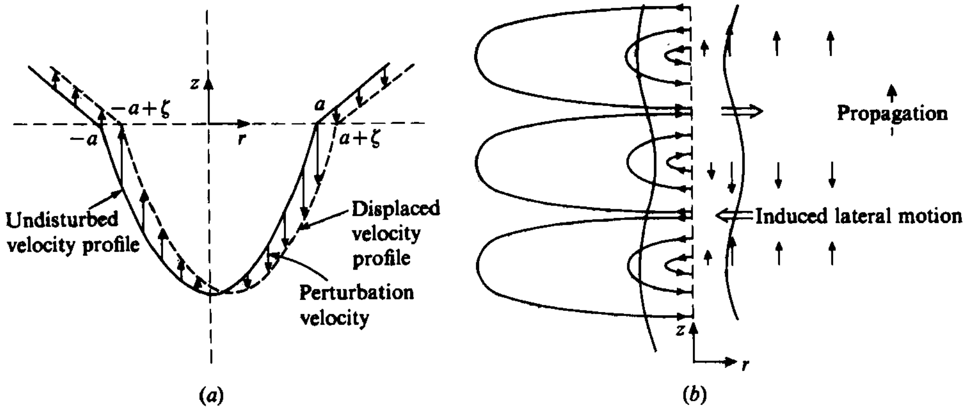

FIGURE 5. (a) In the limit $R \to 0$, a meandering displacement of the interface and basic-flow profile causes a vertical perturbation velocity. (b) The variation of the vertical perturbation velocity in (a) causes horizontal motion and propagation of the disturbance upwards.

5.2. *Meandering perturbations*

The solution of (5.3a) subject to the boundary conditions (5.3b, d) is

$$\psi_0 = (A_0 \alpha q + B_0)\,\mathrm{e}^{-\alpha q} \qquad (r > 1), \tag{5.10a}$$

$$\psi_0 = C_0 \alpha r \sinh(\alpha r) + D_0 \cosh(\alpha r) \qquad (r < 1). \tag{5.10b}$$

As in §5.1, the substitution of ψ_0 from (5.10) into the interfacial-jump conditions leads to linear homogeneous equations for the unknown constants A_0, B_0, C_0 and D_0. The determinant of the matrix of coefficients must vanish for a non-trivial solution to exist. This implies that

$$c_0 = \frac{\left(\dfrac{1}{2\alpha} + \beta - 1\right) - \tfrac{1}{2}\mathrm{i}\Gamma[C^2 + (CS+\alpha)\beta]}{(C^2+S^2) + (CS+\alpha)\beta + (CS-\alpha)\beta^{-1}}. \tag{5.11}$$

For long-wavelength disturbances

$$c_0 \doteq \frac{1}{2\alpha} - \tfrac{1}{2}\mathrm{i}\Gamma, \tag{5.12}$$

corresponding to a wave propagating against the direction of plume flow and stabilized by surface tension. We note that surface tension is much more effective at stabilizing meandering than varicose perturbations. Detailed analysis shows that this stabilizing effect cannot be overcome by the destabilizing influence of inertia when $R \ll \alpha^3$ and $\Gamma = O(1)$.

The leading-order stream function when $\alpha \ll 1$ is given by

$$\psi_0 \approx A(1 - \alpha(r-1))\,\mathrm{e}^{-\alpha(r-1)} \qquad (r > 1), \tag{5.13a}$$

$$\psi_0 \approx A(1 - \beta\alpha r^2) \qquad (r < 1). \tag{5.13b}$$

We see that a meandering displacement of the interface to $r = \pm a + \zeta$ causes a perturbation flow that shifts the parabolic mean-flow profile centred at $r = 0$ towards one centred at $r = \zeta$ (see figure 5a). The effect is to cause vortical motion at the points of maximum interfacial displacement and lateral flow at the points of zero displacement. This lateral flow is in the appropriate direction to cause propagation of the disturbance upwards (see figure 5b). This agrees with (5.12).

Mode	Limit	Growth rate	Number	Stability
Varicose	$\alpha \to 0$	$c = O[(\alpha R)^{-\frac{1}{3}}]$	∞	one unstable, rest stable
Meander	$\alpha \to 0$	$c = O[(\alpha R)^{-\frac{1}{5}}]$	∞	all stable
Varicose	$\alpha \to 0$	$c = O(\alpha^{-1})$	none	—
Meander	$\alpha \to 0$	$c \sim \alpha^{-1}(1-\beta)$	1	unstable at higher order if $\nu_0 < \beta_c(R)\,\nu_i$
Varicose	$R \to 0, \quad \alpha \ll 1$	$c \sim -(2\alpha)^{-1}$	1	unstable if $R > \frac{8}{3}\Gamma\alpha^5$
Meander	$R \to 0, \quad \alpha \ll 1$	$c \sim (2\alpha)^{-1}$	1	stable

TABLE 1. Classification of modes of perturbation

6. Conclusions

We have examined the linear stability of a steady plume of buoyant fluid, considering throughout the situation in which the boundaries of the external fluid are at a much greater distance from the plume than the wavelength of the disturbance. Expressions for the growth rate c as a function of the relevant dimensionless parameters in a variety of asymptotic limits are summarized in table 1.

The results show that a steady line plume is always unstable to long-wavelength varicose disturbances. Such a plume is also unstable to long-wavelength meandering disturbances if the external-fluid viscosity is less than a certain multiple of the internal-fluid viscosity. A further varicose mode of instability exists in the limit $R \to 0$ when surface tension is weak.

These results can be compared with the stability of a three-dimensional plume, determined experimentally by Huppert *et al.* (1986). They found that at low Reynolds numbers an axisymmetric plume seemed stable, at higher Reynolds numbers long-wavelength meandering or varicose instabilities appeared and at large Reynolds numbers unsteadiness and turbulence set in. Our linear stability analysis is in only partial agreement with these experimental results. We have shown that a line plume is unstable at moderate Reynolds numbers to long-wavelength meandering and varicose perturbations. Such perturbations will grow to finite amplitude, and at sufficiently large Reynolds numbers we would expect nonlinear effects to lead to turbulence. Thus far is in agreement with experiment. At low Reynolds numbers, however, we have found a line plume to be unstable to varicose disturbances. Instability at low Reynolds numbers has been found in other Poiseuille or Couette flows with a viscosity discontinuity and small surface tension (Yih 1967; Hickox 1971; Hooper & Boyd 1983; Hooper 1985) and seems to be a feature of such systems. We would, therefore, expect the mechanisms of instability of a line plume to be applicable to the axisymmetric plume. This prediction of instability seems at variance with the experimental observations. In any practical application, however, the plume will rise in a container of finite vertical extent and it is necessary to consider whether the growth rate is large enough for the instability to become apparent during the rise time of the plume. Further work is needed to decide whether the experimentally determined stability of an axisymmetric plume is simply a consequence of the use of apparatus with a finite depth.

It is clearly of interest to seek analogous analytical results for a three-dimensional axisymmetric plume. The undisturbed flow and axisymmetric plume radius are easily calculated. In contrast with the case of a line plume, the plume radius is unaltered at leading order by the presence or absence of a return flow in the outer fluid. The stability analysis of such a flow is difficult; complications arise because the external velocity profile is logarithmic rather than linear and because separation into cylindrical polar coordinates introduces differential operators with non-constant coefficients. We have been unable to complete a solution to this problem, though we expect the mechanisms of instability outlined in this paper to carry over to the axisymmetric case.

The complexity of the analysis makes it difficult to identify the causes of instability. Our results depend only weakly on the density ratio between the fluids, indicating that the fundamental causes of instability must be the viscosity contrast between the fluids and the change in profile of the mean flow across the interface. In the limit $R \rightarrow 0$ we showed that the perturbation velocity is such as to adjust the mean-flow profile to that appropriate to the local plume thickness and location. We then argued that the propagation and growth of such disturbances can be explained in terms of this perturbation velocity. In the limit $\alpha \rightarrow 0$ we showed that the propagation of one mode depends on the difference in shear rates across the interface and hence on the viscosity difference between the fluids. The scaling of the remaining modes of perturbation in powers of $\alpha^{\frac{1}{3}}$ suggests that they may be related to general shear instabilities.

From a mathematical viewpoint, the results display some interesting and surprising features. At long wavelengths we have found a variety of modes of perturbation with values of c that can be $O(\alpha^{-1})$ or $O(\alpha^{-\frac{1}{3}})$. The number of such modes can be infinite, finite or zero, depending on the limit taken, indicating that the eigenvalue problem determining c is subject to extensive bifurcation. In the limit $\alpha \rightarrow 0$, $R = O(1)$ there are two completely different meandering solutions in which the balances between the physical processes of diffusion, advection and propagation occur in different regions.

All these factors indicate that the dispersion relation, $c = c(\alpha, R, \beta, m, \Gamma)$, is highly complicated and the existence of even more modes should not be ruled out. Finally, we would like to comment that the methods of this paper are applicable to other two-fluid flows with distant boundaries. In such cases we would expect similarly varied and exotic behaviour. Recent studies of Couette flow (Hooper & Boyd 1983; Hooper 1986) lend support to this view.

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